Group Theory 9

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Q> If $S \leq G$ and $[G:S] = 2$ then $S \triangleleft G$. Prove it.

Ans:- One Way $G = S \sqcup aS$ .. $a \notin S$

$e \in S$, $geg^{-1} = e \in gSg^{-1}$

$e \notin aS$

$gSg^{-1} \supset S$

Second way

$G = (S) \sqcup (G-S)$

If $x \in S \Rightarrow xS = S = Sx$

If $x \in G-S \Rightarrow xS \neq S$, so, $xS = G-S = Sx$

Right coset = Left coset $\Rightarrow$ S is a normal group

Q> Let H and D be subgroups of G Prove that $H \cap D$ is subgroup of G as well

Q> Let D be a set of all elements of finite order in an abelian group G Prove that D is a subgroup of G

Ans:- $D = \{ a : a^n = e \text{ for some } n \}$

$e \in D$

$a, b \in D \Rightarrow a^m = e$, $b^l = e$ for some $m, l \in \mathbb{N}$

$(ab)^{ml} = a^{ml} b^{ml} = e$ Ord(a) = m, Ord(b) = l

$\Downarrow$

$\Rightarrow ab \in D$ Ord$(a^{-1}) = m$

$a^{m-1} = a^{-1} \in D$ $(a^{-1}b)^{ml} = e$

So D is a subgroup

Quotient Groups :-

Theorem:- If $N \trianglelefteq G$, then the cosets of N in G, denoted by G/N is of order $[G:N]$

Proof - $G/N = \{aN \text{ such that } a \in G\}$

$N = eN$

$aN\, bN = abN$

$(aN)^{-1} = a^{-1}N$

$|G/N| = [G:N]$

Definition:- The group G/N is the quotient group

•> If $N \trianglelefteq G$ then the natural map, $f: G \to G/N$, $a \to Na$

is a surjective homomorphism with kernel N

Definition:- If $a, b \in G$ the commutator of a and b is denoted by $[a, b]$ i.e, $[a\,b] = aba^{-1}b^{-1}$

The commutator subgroup of G is the subgroup generated by all the commutators

Theorem:- The commutator subgroup G' is a normal subgroup of G. If $H \trianglelefteq G$ then G/H is abelian iff $G' \leq H$

Q) Let G be a finite group of odd order and let x be the product of all elements of G in some order. Prove

Q) Let G be a fini[illegible] Prove

the product of all elements of G in some order

that $x \in G'$

Ans:- $x = \prod_{a \in G} (a) = 1$ if G is abelian $\Rightarrow x \in G'$

If G not abelian, $x = \prod_{a \in G} (a)$, G/G' is abelian

$b = aG' \in G/G'$ $\qquad |gG'| = |G'| \; \forall g \in G$

$$\Rightarrow xG' = \left(\prod_{a \in G} a\right) G' = \left(\prod_{b \in G/G'} b\right)^{|G'|} \in G/G'$$

$$\Rightarrow xG' \in G/G' \Rightarrow x \in G'$$

Isomorphism Theorems:-

$\boxed{\text{Theorem:- (First Isomorphism Theorem)}}$

Let $f : G \to H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong im(f)$

Proof:- $\varphi : G/K \to H$ $\qquad \Rightarrow im(\varphi) = im(f)$

$Ka \to f(a)$ $\qquad \varphi$ is also injective

$\varphi(Ka) = \varphi(Kb) \Rightarrow Ka = Kb$

$\Rightarrow f(a) = f(b)$ $\qquad f(ab^{-1}) = 1$, $ab^{-1} \in K$

So, φ is an isomorphism

$\varphi^{-1} : im(f) \to G/K$

$\varphi^{-1}(x) = Ka$ $\qquad$ where $f(a) = x$

$\varphi^{-1}(x') = Ka'$ $\qquad$ where $f(a') = x'$

$\varphi^{-1}(x) = Ka$

$\varphi^{-1}(x') = Ka'$ where $f(a') = x'$

If $\varphi^{-1}(x) = \varphi^{-1}(x') \Rightarrow Ka = Ka' \Rightarrow f(a) = f(a')$

So we get that, φ^{-1} is also injection

$$G/K \cong \operatorname{im}(f)$$

<u>Lemma</u>:- If S and T are subgroups of G and if one of them is normal then $ST = TS$

Ans:- WLOG, Let S be normal,

$s \in ST$ then, $s = s_1 t_1$ for some $s_1 \in S$, $t_1 \in T$

$$s_1 = t_1 s_1 t_1^{-1}$$

$$s = t_1 s_1 t_1^{-1} t_1 = t_1 s_1 \in TS$$

$$ST \subset TS, \quad TS \subset ST \Rightarrow TS = ST$$